TURBULENT FLOW OF A POLYMER SOLUTION OVER A FLAT PLATE

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Logarithmic drag equations are derived by the Prandtl—Kármán method. Error estimates for certain simplifications are given, and errors occurring in the literature are eliminated. The maximum drag-reduction regime is discussed.

The simplest method of analyzing a turbulent boundary layer on a plate in a viscous fluid is the Prandtl—Kármán method [1, 2], which is based on the notion of a local dependence of the average velocity on the dynamic velocity $u_*(x)$ and boundary layer thickness $\delta(x)$ and the notion of coincidence of the velocity distributions near a plate and near the walls of a pipe.

This method has been extended to the description of a boundary layer of polymer solutions [3-5]. However, White [3] commits a fundamental error, while the other two authors [4, 5] use formal expansions without any error discussion. We now show that simple drag equations are obtained by virtue of the small parameters in the problem.

1. Prandtl—Kármán Method for Polymer Solutions

In fully developed turbulent flow far from the surface of the plate large-scale inertial motions play a dominant role without being directly affected by the polymer at the small concentrations in question.† The average velocity distribution can be described by the usual "velocity defect law" [1]

$$V^+ \equiv V/u_* = \langle u^+ \rangle + f_1(z|\delta), \qquad z > \Delta_2. \tag{1.1}$$

Here only the position Δ_2 of the boundary of the wall buffer zone is affected by the presence of the polymer.

If the layer thickness Δ_2 is relatively small ($\Delta_2/\delta \ll 0.1$), we obtain from the equations of motion and incompressibility (density $\rho = 1$), neglecting‡ the contribution of the zone $z < \Delta_2$ and normal stresses, the Kármán integral relation

$$V^{+2}\frac{\partial \delta_2}{\partial x} = 1, \quad V^{+2}\delta_2 = (D_1V^+ - D_2)\,\delta, \quad D_n = \int\limits_0^1 f_1^n(\eta)\,d\eta, \tag{1.2}$$

the form of which is also independent of the type of fluid.

The universal function $f_1(\eta)$ can be approximated with acceptable accuracy by the expression (see Gorodtsov [6])

$$f_1(\eta) = \begin{cases} C_1(1-\eta)^2, & 1 > \eta > \eta_1, \\ -A_1\ln\eta + B_1, & \eta_1 > \eta > \Delta_2/\delta, \end{cases} \tag{1.3}$$

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† It is assumed that the contribution of the forward part of the plate with undeveloped turbulence to the drag is negligible (large plates and velocities).

‡ The case in which $\Delta_2 \approx 0.15\,\delta$ and in which it is necessary to consider the zone $z < \Delta_2$ will be discussed in Sec. 5.

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where $A \approx 2.44$, $C_1 \approx 9.6$ [1, 2], and the conditions for smooth matching at the point $\eta = \eta_1$ are $\eta_1 \approx 0.15$ and $B_1 \approx 2.3$. In this approximation $D_1 \approx 3.4$ and $D_2 \approx 22$.

The magnitude of the velocity is sensitive to the presence of a dissolved polymer. According to studies of flows of solutions in large pipes [7, 8], even though for $z > \Delta_2$ the velocity is a logarithmic function of z (as without the polymer), it increases by ΔB. Here ΔB depends on the characteristics of the solution, and this dependence can be regarded with sufficient accuracy as involving two parameters (β and u_{*cr}):

$$\Delta B = \beta A_1 \ln (V_{cr}/V^+)\, h(V_{cr} - V^+). \tag{1.4}$$

In the latter expression h is the unit Heaviside function, which has a nonzero value only for $V^+ < V_{cr} \equiv V/u_{*cr}$.

According to the fundamental postulate of the Prandtl–Kármán method Eq. (1.4) can also be used to describe the flow of a solution over a plate.

The existence of a zone with a logarithmic velocity profile is known [1] to be attributable to overlapping of the zones in which the outer and inner similarity laws hold, and

$$V^+ = A_1 \ln \delta^+ + B_0 + B_1 + \Delta B, \qquad \delta^+ \equiv u_*\delta/\nu. \tag{1.5}$$

In viscous fluids $B_0 = 5.6$, $\Delta B = 0$, $\Delta_2^+ \approx 30$, and the condition for existence of a logarithmic zone $\Delta_2 < \eta_1\delta$ (noncontiguity of the wall and outer transition zones), or $\delta^+ > 200$, reduces by virtue of (1.5) to the requirement

$$V^+ > 20. \tag{1.6}$$

In polymer solutions Δ_2^+ increases, and condition (1.6) is all the more necessary.

Besides the large parameter V^+, it will be advantageous below to use the small parameter $\varepsilon(V^+)$:

$$\frac{\partial \ln \delta_2}{\partial \ln V^+} = \frac{V^+}{A_1} + \beta + \frac{D_2}{D_1 V^+ - D_2} \equiv \frac{1}{\varepsilon(V^+)} \approx \frac{V^+}{A_1} + \beta + 0.35. \tag{1.7}$$

Here, according to (1.6), we have $\varepsilon^{-1} > 8.7 + \beta$, and the approximate equality is satisfied with less than two-percent error.

Consequently, for the two unknown functions $u_*(x)$ and $\delta(x)$ we have a complete system of equations (1.2), (1.4), (1.5) containing small parameters. They are not exactly integrable, in general, except in the cases of a viscous fluid ($\beta = 0$) and integer values of β. The opposite conclusion is drawn by White [3] due to a gross error. Landweber and Poreh [9] use the integrability for integer-valued β in their calculations. In the general situation, due to the small parameters, approximate integration is possible and has been carried out [4] by means of formal expansions in reciprocal powers of V^+. The expansions, however, include terms with powers of the parameter $\beta A_1/V^+$ [such terms arise, for example, in the series expansion of $\varepsilon(V^+)$ in reciprocal powers of V^+], which can be close to unity. Additional precision is therefore needed.

Integrating by parts, we obtain from (1.2) and (1.7)

$$\begin{gathered} x - x_0 = I_0 \equiv \int V^{+2} d\delta_2 = (1 - 2\varepsilon + 2\varepsilon^2) V^{+2}\delta_2 + I_1, \\ I_1 = \int (1 - 2\varepsilon)\varepsilon^3 \left(\varepsilon^{-1} - \frac{\partial \varepsilon^{-1}}{\partial \ln V^+}\right) V^{+2} d\delta_2, \end{gathered} \tag{1.8}$$

where by inequality (1.6) we have the upper bound $I_1 < [(\beta + 1.2)/(\beta + 8.7)^3] I_0$. This bound permits us to rewrite (1.8) as follows with less than one-percent error:

$$x - x_0 = V^{+2}\delta_2 \exp(-2\varepsilon). \tag{1.9}$$

Thus, the differential equation $(1.2)_1$ is replaced with acceptable accuracy by the algebraic equation (1.9). All that remains is to transform the system of algebraic equations $(1.2)_2$, (1.4), (1.5), (1.7), and (1.9) to a useful form.

In the flow of a homogeneous polymer solution over a smooth plate the stress on the wall decreases as the turbulent boundary layer develops [see (2.7)], and over the "length" of the plate ($L > L_{cr}$) it can fall below the critical stress u_{*cr}^2 for influence of the polymer. Beginning with $x = L_{cr}$, the polymer no longer affects the turbulence.

2. "Small" Plate ($L < L_{cr}$)

On a short plate the stress on the wall is above the critical level everywhere, and the system of equations $(1.2)_2$, (1.4), (1.5), (1.9) can be written in the form

$$\mathrm{Re}_2 \equiv V\delta_2/\nu = \left(D_1 - \frac{D_2}{V^+}\right)\left(\frac{V^+}{V_{\mathrm{cr}}}\right)^{\beta} \exp\left[\frac{V^+ - B_0 - B_1}{A_1}\right], \tag{2.1}$$

$$\mathrm{Re}_x \equiv Vx/\nu = Vx_0/\nu + V^{+2}\,\mathrm{Re}_2 \exp[-2\varepsilon(V^+)]. \tag{2.2}$$

Here x_0 is determined by the initial conditions for development of a turbulent boundary layer. Near the point of its origin the wall stress is a maximum, and the polymer necessarily exerts maximum influence on the nature of the transition from laminar to turbulent flow. To date, however, this problem has not been studied experimentally for flow of a solution over a plate. † We assume hereinafter that $\mathrm{Re}_x \gg Vx_0/\nu$ and often drop the term containing x_0.

Eliminating Re_2 and introducing the local friction coefficient $c_f \equiv 2/V^{+2}$, we rewrite (2.1) and (2.2) in the conventional form

$$\sqrt{2/c_f} = A_1 \ln(\mathrm{Re}_x c_f/2) + \beta A_1 \ln(V_{\mathrm{cr}}\sqrt{c_f/2}) + B_0 + B_1', \tag{2.3}$$

$$B_1' = B_1 - A_1 \ln(D_1 - D_2\sqrt{c_f/2}) + 2A_1\varepsilon(\sqrt{c_f/2}). \tag{2.4}$$

Within the one-percent error limits applicable in the derivation of Eqs. (2.3) and (2.4) we can approximate B_1' by the expression $-0.7 + 20\sqrt{c_f}$ and deduce the drag equation‡ ($\alpha \equiv 5.6\beta$)

$$1/\sqrt{c_f} = 4\lg(\mathrm{Re}_x c_f) + \frac{\alpha}{\sqrt{2}}\lg(V_{\mathrm{cr}}\sqrt{c_f/2}) + 2.3 + 14\sqrt{c_f}. \tag{2.5}$$

If we allow a maximum three-percent error, we can put $B_1' \approx 1/3$:

$$1/\sqrt{c_f} = 4\lg(\mathrm{Re}_x c_f) + \frac{\alpha}{\sqrt{2}}\lg(V_{\mathrm{cr}}\sqrt{c_f/2}) + 3. \tag{2.6}$$

The implicit nature of the dependence of c_f on Re_x is a drawback of these equations. However, owing to the smallness of the parameter $\sqrt{c_f/2} = 1/V^+$ they can be approximately expanded in V^+. Thus, in the interval of practical interest $20 < V^+ < 120$, replacing $\log V^+$ by $1.2 + 0.08\,V^+$, we can transform the equation as follows within an error of a few percent:

$$(1 + 0.04\beta)\frac{1}{\sqrt{c_f}} = 3.6\lg[\mathrm{Re}_x(V_{\mathrm{cr}}/16)^{\beta}] - 5. \tag{2.7}$$

It is clear from this result that the stress on the wall diminishes monotonically along the plate.

3. "Large" Plate ($L > L_{cr}$)

Up to the point $x = L_{cr}$ the stress on the wall exceeds the critical value, and the development of the boundary layer is the same as for a "small" plate. After this point the boundary layer develops as in the case of a viscous fluid.

Taking $x = L_{cr}$, $V^+ = V_{cr}$ as the starting point for the integration of Eq. $(1.2)_1$, we obtain equations of the type (2.1) and (2.2) with β and x_0 replaced by, respectively, 0 and x_{cr}:

$$x_{\mathrm{cr}} = L_{\mathrm{cr}} - \frac{\nu}{u_{*\mathrm{cr}}}(D_1 V_{\mathrm{cr}} - D_2)\exp\left[\frac{V_{\mathrm{cr}} - B_0 - B_1}{A_1} - \frac{2A_1}{V_{\mathrm{cr}}}\right]. \tag{3.1}$$

Inasmuch as Eqs. (2.1) and (2.2) hold for $x \le L_{cr}$, they can be used to express L_{cr} in terms of V_{cr} and β:

$$L_{\mathrm{cr}} = x_0 + \frac{\nu}{u_{*\mathrm{cr}}}(D_1 V_{\mathrm{cr}} - D_2)\exp\left[\frac{V_{\mathrm{cr}} - B_0 - B_1}{A_1} - 2\varepsilon(V_{\mathrm{cr}})\right] \tag{3.2}$$

Hence we infer that

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† In "small" pipes the so-called "early turbulence" effect is observed in the transition region.

‡ Other numerical values of the coefficients A_1 and B_0 are frequently chosen in the drag equations to provide a better fit with the experimental data [2].

$$x_{\mathrm{cr}} \approx x_0 + \frac{\nu}{V} \frac{2\beta A_1^2}{V_{\mathrm{cr}} + \beta A_1} (D_1 V_{\mathrm{cr}} - D_2) \exp\left[\frac{V_{\mathrm{cr}} - B_0 - B_1}{A_1} - \frac{2A_1}{V_{\mathrm{cr}}}\right], \tag{3.3}$$

where the second term is less than $0.1\,L_{cr}$. For $x > L_{cr}$, therefore, the quantity x_{cr}, like x_0 previously, can be dropped.

Consequently, for $\mathrm{Re}_x < VL_{cr}/\nu$ the equation for the local friction coefficient of a long plate in a polymer solution has the form (2.5), and for $\mathrm{Re}_x > VL_{cr}/\nu$ it is the same as for a viscous fluid (i. e., for $\beta = 0$):

$$1/\sqrt{c_f} = 4\lg(\mathrm{Re}_x c_f) + 2.3 + 14\sqrt{c_f}. \tag{3.4}$$

Within one-percent error limits this relation can be rewritten in the form [see (2.7)]

$$1/\sqrt{c_f} = 3.35\lg \mathrm{Re}_x - 3. \tag{3.5}$$

4. Total Friction Coefficient

From Eq. $(1.2)_1$ we obtain, assuming that $\delta_2(0) \ll \delta_2(L)$,

$$C_f \equiv \frac{1}{L}\int_0^L c_f(x)\,dx = \frac{2\delta_2(L)}{L} = \frac{2}{\mathrm{Re}} \frac{V\delta_2(L)}{\nu}, \tag{4.1}$$

and now from Eqs. (2.1) and (2.2) we can deduce parametric equations relating the total friction coefficient C_f and Reynolds number $\mathrm{Re} \equiv VL/\nu$:

$$C_f = \frac{2}{\xi^2}\exp[2\varepsilon(\xi)], \qquad \xi = V/u_*(L), \tag{4.2}$$

$$\mathrm{Re} = \xi(D_1\xi - D_2)\left(\frac{\xi}{V_{\mathrm{cr}}}\right)^{\beta} \exp\left[\frac{\xi - B_0 - B_1}{A_1} - 2\varepsilon(\xi)\right]. \tag{4.3}$$

Within one-percent error limits relation (4.2) is readily expanded in the parameter ξ:

$$\xi = \sqrt{\frac{2}{C_f}} \exp\left[\varepsilon\left(\sqrt{\frac{2}{C_f}}\right)\right] \approx \sqrt{\frac{2}{C_f}}\left(1 + \frac{A_1}{\sqrt{2/C_f} + \beta A_1}\right), \tag{4.4}$$

and the substitution of (4.4) into (4.3) yields an expression for Re in terms of C_f. On the other hand, Eq. (4.4) enables us to determine the total friction coefficient C_f from the known local friction coefficient $c_{fL} \equiv c_f(L)$:

$$C_f = c_{fL}\left(1 + \frac{3.5\sqrt{c_{fL}}}{1 + 1.7\beta\sqrt{c_{fL}}}\right). \tag{4.5}$$

Eliminating the parameter ξ from (4.2) and (4.3) with one-percent error, we obtain the drag equation in its traditional form:

$$1/\sqrt{C_f} = \frac{A_1}{\sqrt{2}}\ln(\mathrm{Re}\,C_f) + \beta\frac{A_1}{\sqrt{2}}\ln(V_{\mathrm{cr}}\sqrt{C_f/2}) + 0.7 + 8.5\sqrt{C_f}. \tag{4.6}$$

An expression of this type has been obtained earlier [4] for polymer solutions and, as already noted by Landweber and Poreh [9], is satisfactory, despite the inadequacy of the intermediate calculations in the former paper.

The inadequacy of Granville's calculations [4], as mentioned, lies in the fact that the expansions include terms containing the parameter $\beta A_1\sqrt{C_f/2}$, which cannot be small. Such terms, however, cancel one another in the final stage of the transition to (4.6).

If, rather than the freestream velocity V, the plate length L is taken as the fixed parameter, it is necessary in the drag equation to make the substitution $V = \mathrm{Re}\nu/L$, or $V_{cr} = \mathrm{Re}\nu/(Lu_{*cr})$; see Gorodtsov [10] for details.

In a viscous fluid the quantity $1/\sqrt{C_f}$ essentially varies from 14 to ~ 30, and (4.6) can be simplified, replacing

$$\ln\sqrt{C_f} + 5\sqrt{C_f} \quad \text{by} \quad -3.4 - 0.11\sqrt{C_f}:$$

$$1/\sqrt{C_f} = 3.3\lg \mathrm{Re} - 4.5. \tag{4.7}$$

5. Maximum Drag Reduction

We have assumed up to now that the thicknesses of the viscous sublayer and wall buffer zone are negligible in comparison with the thickness of the boundary layer. We now examine the case in which the buffer zone is contiguous with the outer mixing zone of the flow, i. e., $\Delta_2 \approx 0.15\delta$. We assume that this flow regime, for which the large-scale structure of the outer flow zone is the only factor left unaltered by addition of the polymer, is the maximum drag-reduction regime.†

It is well known [8] that with a large drag reduction the velocity profile in the buffer layer $\Delta_1 < z < \Delta_2$ is close to logarithmic, so that it is once again permissible to use an approximation of the form (1.3) with Δ_1/δ taken as the lower bound. However, if $C_1 \approx 9.6$ and $\eta_1 \approx 0.15$ remain unchanged due to the presumed invariance of the outer transition zone, the remaining coefficients‡ now have different numerical values:

$$A_1 = 11.6;\quad B_0 = -16.8;\quad B_1 = -15.1;\quad D_1 = 4.7;\quad D_2 = 80. \tag{5.1}$$

In the given special case, as for a viscous fluid, the integral I_0 in (1.8) can be computed exactly (these two cases differ only in the numerical coefficients):

$$I_0 = \left(1 - \frac{2A_1}{V^+} + \frac{2D_1A_1^2}{D_1V^+ - D_2}\,\frac{1}{V^+}\right)V^{+2}\delta_2, \tag{5.2}$$

and the equations are reduced to algebraic expressions [cf. (2.1) and (2.2)]:

$$\mathrm{Re}_2 = \left(D_1 - \frac{D_2}{V^+}\right)\exp\left[\frac{V^+ - B_0 - B_1}{A_1}\right], \tag{5.3}$$

$$\mathrm{Re}_x = \left[1 - \frac{2A_1}{V^+} + \frac{2D_1A_1^2}{V^+(D_1V^+ - D_2)}\right]V^{+2}\,\mathrm{Re}_2. \tag{5.4}$$

These relations, in turn, can within one-percent error limits be rewritten in the traditional form for the drag equation:

$$1/\sqrt{c_f} = 8.2\ln(\mathrm{Re}_x c_f) - 41 - 8.2\ln(1 - 29\sqrt{c_f} + 300c_f). \tag{5.5}$$

For the total friction coefficient C_f, using (4.1), (5.3), and (5.4), we write

$$C_f = \frac{c_{fL}}{1 - 16.4\sqrt{c_{fL}}\,[1 - 8{,}2/(\sqrt{c_{fL}} - 12)]}. \tag{5.6}$$

Finally, from (4.1), (5.3), and (5.4) we obtain in approximate fashion the drag equation (cf. Granville [12])

$$1/\sqrt{C_f} = 8.2\ln(\mathrm{Re}\,C_f) - 49 + 140\sqrt{C_f}. \tag{5.7}$$

Note that for $\mathrm{Re} > 10^6$ the error incurred in the transition from (5.3)-(5.4) to (5.7) is a few percent, while for smaller Reynolds numbers the error increases to ten percent.

NOTATION

δ, boundary layer thickness; δ_2, momentum-loss distance; Δ_2, thickness of wall buffer zone; V, free-stream velocity; u_*, dynamic velocity; u_{*cr}, critical dynamic velocity at which influence of the polymer begins; β, dimensionless characteristic of the intensity of influence of the polymer; Re, Reynolds number; c_f, C_f, friction coefficients; A_1, B_0, B_1, C_1, D_1, D_2, numerical coefficients.

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† Landweber and Poreh [9] invoke the same assumption. It is postulated in other papers [11, 12] that turbulence can also be influenced by polymers in the wall region of a pipe and at the outer edge of the boundary layer.

‡ The matching at the point $\eta = \eta_1$ is no longer smooth.

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FLUID FRICTION OF A POLYMER SOLUTION FLOWING IN A LARGE-DIAMETER PIPE

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Experimental data are given from comparative tests to determine the fluid friction in flows of a Polyox solution with a concentration $c = 7 \cdot 10^{-6}$ g/cm^3 in pipes having diameters $d = 35.5$ mm and $d = 514$ mm.

§1. The discovery of drag reduction effected in turbulent flows of water near a rigid wall by the addition of small quantities of high-molecular-weight compounds (polymers) to the flow has in the last few years motivated extensive research aimed at explaining this phenomenon and devising practical methods for predicting the attainable net effect. One of the possible techniques for calculating the net effect of drag reduction in pipe flows of polymer solutions has been proposed by the authors [1]. The method is based on universal graphs of the investigated influence of polymer additives as a function of the type of polymer, flow velocity in the pipe, and concentration of the solution [1, 2]. However, all the experimental material used for analysis and plotted in the form of universal graphs refers to flows in pipes whose diameters do not exceed 35 mm and the flow velocity is such that the range of Reynolds numbers is $7 \cdot 10^3$ to $3 \cdot 10^5$. The published data on the influence of polymer additives in a flow on the friction in pipes have also been obtained in the same range of pipe diameters ($d < 50$ mm) and Reynolds numbers ($\mathrm{Re} < 5 \cdot 10^5$) and correspond qualitatively to the results of our earlier generalization [1, 2].

Thus, all the cited experiments have been conducted under conditions of a limited range of Reynolds numbers in comparatively small-diameter pipes. The difficulties inherent in the experimental investigation of the characteristics of turbulent flow of polymer solutions in pipes of large diameter stem primarily from the

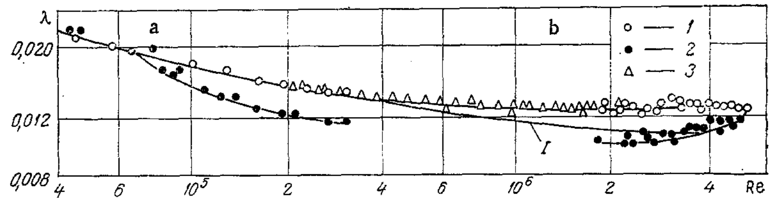

Fig. 1. Coefficient of fluid friction versus Reynolds number Re for flow of water and a Polyox solution with concentration $c = 7 \cdot 10^{-6}$ g/cm^3 in pipes with diameters $d = 35.5$ mm (a) and $d = 514$ mm (b). 1) Water; 2) Polyox solution; 3) Galavics' tests with water [4]: I) $1/\sqrt{\lambda} = 2 \log \mathrm{Re}\sqrt{\lambda} - 0.8$.

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